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Estimates for the Discrepancy of a Signed Measure Using Its Energy Norm

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W. Kleiner established (1964, *Ann. Polon. Math.* **14**, 117–130) for smooth curves and arcs an estimate for the discrepancy of a signed measure by using its energy norm. We extend this result to quasiconformal curves and arcs. The proof uses the theory of quasiconformal mappings and condenser theory. In a first step, the discrepancy of a signed measure can be estimated from above in terms of its energy norm and the capacity of a special condenser. This result is valid on every Jordan curve. Examples show the sharpness of the results from various points of view.

1. INTRODUCTION

Let $E \subset \mathbb{C}$ be a bounded Jordan curve or Jordan arc and let σ be a signed Borel measure supported on E. The discrepancy of σ is defined by

$$D[\sigma] := \sup |\sigma(J)|,$$

where the supremum is taken over all subarcs $J \subseteq E$. If $\{v_n\}$ is a sequence of Borel measures on E converging to a Borel measure μ in the sense that $D[\mu - v_n] \to 0$ as $n \to \infty$, then $\{v_n\}$ converges to μ in the weak-star sense. Thus, the discrepancy between μ and v_n defined by $D[\mu - v_n]$ serves as a measurement on the rate of the weak-star convergence.

In applications, frequently $\mu = \mu_E$ is the equilibrium measure of E [24, p. 55] and $v_n = v_{p_n}$ is the normalized zero counting measure of a polynomial p_n of degree n, i.e., the measure which associates the mass $1/n$ with each of the zeros of p_n, where every zero is counted according to its multiplicity.

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Usually, the discrepancy is estimated in terms of bounds for the logarithmic potential $U(\sigma, z)$ of σ defined by

$$U(\sigma, z) := \int \log \frac{1}{|z-t|} \, d\sigma(t).$$

Typical results can be found in [4, 5, 7–11, 23].

A completely different approach was chosen by Kleiner [16]. He used the energy norm $\|\sigma\|$ of the signed measure σ, which is defined by

$$\|\sigma\|^2 := \int\int \log \frac{1}{|z-t|} \, d\sigma(t) \, d\sigma(z),$$

to estimate the discrepancy of σ. To state the result of Kleiner some preparing definitions are necessary. Considering the set of measures

$$\mathscr{M}_+(E) := \{\mu : \mu \text{ is a positive measure with } \operatorname{supp}(\mu) \subseteq E\},$$

we introduce a relation on $\mathscr{M}_+(E)$ be setting for $\mu_1, \mu_2 \in \mathscr{M}_+(E)$,

$$\mu_1 \leqslant \mu_2 :\Leftrightarrow \mu_1(J) \leqslant \mu_2(J), \qquad \forall J \subseteq E, \ J \text{ Borel-measurable}.$$

Furthermore let $v \in \mathscr{M}_+(E)$ be a positive measure with continuous logarithmic potential in $\mathbb{C}$. For smooth E we define a modulus of continuity ω of v by

$$\omega(\varepsilon) := \sup\{v(J) : J \text{ a subarc of } E, \ l(J) \leqslant \varepsilon\}, \qquad \forall \varepsilon > 0,$$

where $l(J)$ denotes the arc length for all rectifiable subarcs $J \subseteq E$. The inverse function is introduced by

$$\omega^{-1}(t) := \inf\{l(J) : J \text{ a subarc of } E, \ v(J) \geqslant t\}, \qquad \forall 0 < t \leqslant v(E).$$

Now, for signed measures $\sigma = \sigma^+ - \sigma^-$ with $\sigma^+, \sigma^- \in \mathscr{M}_+(E)$, that are dominated by v such that for their total variation

$$|\sigma| := \sigma^+ + \sigma^- \leqslant v,$$

the discrepancy can be estimated as follows.

THEOREM A (Kleiner [16]). *Let $E \subset \mathbb{C}$ be a smooth Jordan curve or smooth Jordan arc and $v \in \mathscr{M}_+(E)$ with finite energy. For each signed measure $\sigma \not\equiv 0$ supported on E with*

$$|\sigma| \leqslant v, \qquad \sigma(E) = 0,$$

there exist constants $c>0$ and $M<1$ only depending on E such that

$$(D[\sigma])^2 \leqslant c\,\|\sigma\|^2 \log \frac{1}{\omega^{-1}(M\,D[\sigma])}.$$

Based on this theorem and its method of proof Kleiner developed a special technique to obtain quantitative results for the distribution of Fekete points on smooth Jordan curves [17]. This technique was a basic tool to get asymptotic estimates about the distribution of extremal points in Chebyshev approximation [10] and one of the starting points for considerations to derive discrepancy estimates for signed measures if a lower bound for the logarithmic potential is known [5].

It turns out that this interesting result of Kleiner can be extended to a wider class of measures and a wider class of curves and arcs. Since the original proof of Kleiner is very technical and difficult to follow we think that our proof provides a completely different and more natural approach. In particular, the dominating measure ν, which is essential for the proof of Kleiner, isn't needed.

2. MAIN RESULTS

Let $K_1, K_2 \subset \mathbb{C}$ be disjoint compact sets and let $\mathcal{M}^1_+(K_i)$ denote the collection of all unit Borel measures in $\mathcal{M}_+(K_i)$, $i=1, 2$. Introducing

$$\mathcal{M}^1(K_1, K_2) := \{\sigma = \sigma_{K_1} - \sigma_{K_2} : \sigma_{K_1} \in \mathcal{M}^1_+(K_1), \sigma_{K_2} \in \mathcal{M}^1_+(K_2)\},$$

we define the modulus of the condenser (K_1, K_2) by

$$\operatorname{mod}(K_1, K_2) := \inf\{\|\sigma\|^2 : \sigma \in \mathcal{M}^1(K_1, K_2)\}$$

and by

$$\operatorname{cap}(K_1, K_2) := \frac{1}{\operatorname{mod}(K_1, K_2)}$$

its capacity (see [6]). If $\operatorname{mod}(K_1, K_2) < \infty$, there exists a unique measure $\tau = \tau_{K_1} - \tau_{K_2} \in \mathcal{M}^1(K_1, K_2)$ such that

$$\operatorname{mod}(K_1, K_2) = \|\tau\|^2.$$

The measure τ is called the equilibrium measure of (K_1, K_2). If $\overline{\mathbb{C}} \backslash K_1$ and $\overline{\mathbb{C}} \backslash K_2$ are simply connected sets, then $\Omega := \overline{\mathbb{C}} \backslash \{K_1 \cup K_2\}$ is doubly connected. Hence, if K_1 and K_2 are not degenerated to single points, there exists a conformal mapping

$$f\colon \Omega \to A_R := \{z \in \mathbb{C} : 1 < |z| < R\}$$

[14, Chap. V]. Moreover,

$$R = e^{\mathrm{mod}(K_1, K_2)}$$

[24, III.13] and, consequently,

$$\mathrm{cap}(K_1, K_2) = \frac{1}{\log R}.$$

In particular, if ∂K_1 and ∂K_2 are Jordan curves or Jordan arcs, then f^{-1} can be extended continuously to a function $f^{-1}\colon \bar{A}_R \to \bar{\Omega}$. To state some needed properties of the equilibrium measure $\tau = \tau_{K_1} - \tau_{K_2}$, we assume without loss of generality that

$$f^{-1}(\{z \in \mathbb{C} : |z| = R\}) = \partial K_1.$$

If ∂K_1 is a Jordan curve, we have for any subarc $J \subset \partial K_1$,

$$\tau_{K_1}(J) = \frac{1}{2\pi R}\, l(f(J)). \tag{2.1}$$

If ∂K_1 is a Jordan arc, there exist for any subarc $J \subset \partial K_1$ two preimages, i.e., arcs $J', J'' \subset \{z \in \mathbb{C} : |z| = R\}$ such that $f^{-1}(J') = f^{-1}(J'') = J$ and $J' \cap J''$ consists of at most two points. In this case

$$\tau_{K_1}(J) = \frac{1}{2\pi R}\,(l(J') + l(J'')). \tag{2.2}$$

Analogous results are valid for τ_{K_2} (for proofs see [15, Sects. 4.2 and 4.3]).

Now, let E be some Jordan curve and $\sigma = \sigma^+ - \sigma^-$ a signed measure on E with $\sigma(E) = 0$ and finite logarithmic energy. Further, we assume that σ^+ and σ^- have no point mass, i.e., $\sigma^{\pm}(\{z\}) = 0$, $\forall\, z \in E$. Since E is some Jordan curve, we need to introduce a modified modulus of continuity of σ^+. Let

$$\omega_+(\varepsilon) = \omega_+(\sigma^+; \varepsilon) := \sup\{\sigma^+(J) : J \text{ a subarc of } E, \mathrm{diam}(J) \leqslant \varepsilon\},$$

$\forall \varepsilon > 0$, and

$$\omega_+^{-1}(t) = \omega_+^{-1}(\sigma^+; t) := \inf\{\operatorname{diam}(J) : J \text{ a subarc of } E, \sigma^+(J) \geqslant t\},$$

$\forall\, 0 < t \leqslant \sigma^+(E)$.

The main results will use a special partition of E which we introduce first. Defining $m := D[\sigma]$, there exists a closed subarc $J \subset E$, such that

$$\operatorname{diam}(J) \geqslant \tfrac{1}{4} \operatorname{diam}(E), \qquad \operatorname{diam}(E_2) \geqslant \tfrac{1}{4} \operatorname{diam}(E), \tag{2.3}$$

where $E_2 := \overline{E \setminus J}$, and without loss of generality

$$\sigma(J) \geqslant \tfrac{3}{8} m. \tag{2.4}$$

This partition can be obtained as follows: We can choose a subarc $J_1 \subset E$, such that $|\sigma(J_1)| \geqslant \frac{3}{4} m$ and $\operatorname{diam}(J_1) \geqslant \frac{1}{2} \operatorname{diam}(E)$. In selecting two points $z_1, z_2 \in J_1$ with $|z_1 - z_2| = \operatorname{diam}(J_1)$, it is possible to fix a point $z_3 \in J_1$ lying between z_1 and z_2 such that

$$|z_i - z_3| \geqslant \tfrac{1}{2} \operatorname{diam}(J_1), \qquad i = 1, 2.$$

z_3 divides J_1 in two subarcs $J_{1,1}$ and $J_{1,2}$ with

$$\operatorname{diam}(J_{1,i}) \geqslant \tfrac{1}{2} \operatorname{diam}(J_1) \geqslant \tfrac{1}{4} \operatorname{diam}(E), \qquad i = 1, 2.$$

Since $\sigma(E) = 0$, we can assume without loss of generality $\sigma(J_{1,1}) \geqslant \frac{3}{8} m$ (otherwise $\sigma(E \setminus J_{1,1}) \geqslant \frac{3}{8} m$). Hence, $J := J_{1,1}$ has the desired properties (2.3) and (2.4).

Next, since we assume that σ^+ and σ^- have no point mass, we can divide J into three closed subarcs L_1, E_1 and L_2, that are disjoint except for their endpoints, such that

$$0 < \operatorname{diam}(L_1) = \operatorname{diam}(L_2) = \tfrac{1}{2} \omega_+^{-1}(\tfrac{1}{8} m) \leqslant \operatorname{diam}(E_1), \tag{2.5}$$

where L_1 and L_2 include the endpoints of J. By virtue of (2.4) we have

$$\sigma(E_2) \leqslant -\tfrac{3}{8} m, \qquad \sigma^+(L_1) \leqslant \tfrac{1}{8} m \qquad \text{and} \qquad \sigma^+(L_2) \leqslant \tfrac{1}{8} m. \tag{2.6}$$

Finally (2.3) and (2.5) yield

$$\operatorname{diam}(E_1) \geqslant \tfrac{1}{12} \operatorname{diam}(E).$$

Now, we are in position to formulate the first theorem.

THEOREM 1. *Let $E \subset \mathbb{C}$ be a Jordan curve and $\sigma = \sigma^+ - \sigma^-$ a signed measure on E with $\sigma(E) = 0$, finite energy and such that $\sigma^\pm(\{z\}) = 0$, $\forall z \in E$.*

There exist disjoint, closed subarcs E_1 *and* E_2 *of* E *with* $\operatorname{diam}(E_i) \geqslant \frac{1}{12}\operatorname{diam}(E)$, $i=1,2$, *and*

$$\min\{\operatorname{diam}(E_1), \operatorname{diam}(E_2)\} \geqslant \operatorname{diam}(L_i) = \tfrac{1}{2}\omega_+^{-1}(\tfrac{1}{8}D[\sigma]), \qquad i=1,2,$$

where L_1 *and* L_2 *are subarcs with* $L_1 \cup L_2 := \overline{E\setminus\{E_1 \cup E_2\}}$, *such that*

$$D[\sigma] \leqslant c\,\|\sigma\|\,\sqrt{\operatorname{cap}(E_1, E_2)}$$

with an absolute constant $c>0$.

A similar result can be proved for Jordan arcs.

To obtain the desired results that extend Theorem A we give upper and lower estimates for the capacity of the condenser (E_1, E_2) for the case of a quasiconformal curve E.

We recall that, by definition, a K-quasiconformal ($K \geqslant 1$) or briefly quasiconformal curve is the image of the unit circle under some K-quasiconformal mapping $F: \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ [1, 19]. Any subarc of a K-quasiconformal curve is called a K-quasiconformal or briefly a quasiconformal arc.

There exists a geometric characterization of quasiconformal curves [19] and arcs [21]. For example, for curves it can be formulated as follows: E is a quasiconformal curve if and only if there exists a constant $c>0$, depending only on E, such that for $z_1, z_2 \in E$,

$$\min\{\operatorname{diam}(E'), \operatorname{diam}(E'')\} \leqslant c\,|z_1 - z_2|, \tag{2.7}$$

where E' and E'' denote the two arcs of which $E\setminus\{z_1, z_2\}$ consists. Moreover, the constant c and the coefficient of quasiconformality K of E are mutually dependent.

Using this criterion, one can easily verify that convex curves, curves of bounded variation without cusps and rectifiable Jordan curves which have locally the same order of arc length and chord length are quasiconformal. On the other hand, it is of interest to know, that a quasiconformal curve can be everywhere nonrectifiable.

LEMMA 2. *Let* $E \subset \mathbb{C}$ *be a* K*-quasiconformal curve and* E_1, E_2 *disjoint, closed subarcs of* E *with* $\operatorname{diam}(E_i) \geqslant \frac{1}{12}\operatorname{diam}(E)$, $i=1,2$. *Defining subarcs* L_1 *and* L_2 *with* $L_1 \cup L_2 := \overline{E\setminus\{E_1 \cup E_2\}}$ *such that for* $\delta := \operatorname{diam}(L_1) = \operatorname{diam}(L_2)$,

$$\delta \leqslant \min\{\operatorname{diam}(E_1), \operatorname{diam}(E_2)\} \qquad \textit{and} \qquad \delta \leqslant \tfrac{1}{2}\operatorname{diam}(E),$$

there exist constants $C_1, C_2>0$ *only depending on* K, *such that*

$$C_1 \log \frac{k}{\delta} \leqslant \operatorname{cap}(E_1, E_2) \leqslant C_2 \log \frac{k}{\delta} \tag{2.8}$$

with $k := \operatorname{diam}(E)$.

Combining Theorem 1 and Lemma 2 we can prove the following Theorem 3, which gives the desired estimate.

THEOREM 3. *Let* $E \subset \mathbb{C}$ *be a* K*-quasiconformal curve or* K*-quasiconformal arc and* $\sigma = \sigma^+ - \sigma^-$ *be a signed measure on* E *with* $\sigma(E)=0$, *finite energy and such that* $\sigma^{\pm}(\{z\})=0$, $\forall\, z \in E$. *There exists a constant* $c>0$ *only depending on* K, *such that*

$$D[\sigma] \leqslant c\, \|\sigma\| \sqrt{\log \frac{2k}{\omega_+^{-1}((1/8)\, D[\sigma])}}, \tag{2.9}$$

with $k := \operatorname{diam}(E)$.

We would like to remark that the formulation of Theorem 3 is completely independent from the geometric construction that Theorem 1 and Lemma 2 are based on.

Remark. The restriction $\forall\, z$: $\sigma^{\pm}(\{z\})=0$ in Theorem 3 can be removed. Indeed, otherwise we may apply Theorem 3 to the Jordan decomposition $\sigma = \mu^+ - \mu^-$ and use the fact that

$$\omega_+^{-1}(\sigma^+; t) \leqslant \omega_+^{-1}(\mu^+; t).$$

The following theorem shows that inequality (2.9) is sharp up to the constant. In particular, the term including the inverse of the modulus of continuity cannot be removed.

THEOREM 4. *For every* K*-quasiconformal curve* $E \subset \mathbb{C}$ *with (w.l.o.g.)* $\operatorname{diam}(E)=1$ *there exist a constant* $c = c(K)>0$ *and a (non-trivial) family* $(\sigma_\delta)_{0<\delta<\delta_0}$ *of signed measures satisfying the assumptions of Theorem* 3 *such that*

$$\frac{D[\sigma_\delta]}{\|\sigma_\delta\|} \to 0 \qquad \text{as} \qquad \delta \to 0$$

and

$$\frac{D[\sigma_\delta]}{\|\sigma_\delta\|} \geqslant c \sqrt{\log \frac{1}{\omega_+^{-1}((1/8)\, D[\sigma_\delta])}}, \qquad 0<\delta<\delta_0.$$

Finally, we would like to discuss the requirement of quasiconformality in Theorem 3. To this end, we consider a function $g \in C^1[0, 1]$ with

$$g^{(j)}(x) > 0, \qquad \forall\, 0 < x < 1, \tag{2.10}$$

$$\lim_{x \to 0+} g^{(j)}(x) = 0 \tag{2.11}$$

for $j = 0, 1$. In addition, we suppose that g' is Dini-smooth, i.e.,

$$|g'(x_2) - g'(x_1)| < h(x_2 - x_1), \qquad \forall\, 0 \leqslant x_1 < x_2 \leqslant 1, \tag{2.12}$$

where h is an increasing function with

$$\int_0^1 \frac{h(x)}{x}\, dx < \infty.$$

Using g we define a closed Jordan curve E as

$$\begin{aligned} E_1 &:= \{z = x + iy : 0 \leqslant x \leqslant 1,\ y = g(x)\}, \\ E_2 &:= \{z = x + iy : 0 \leqslant x \leqslant 1,\ y = -g(x)\}, \\ E_3 &:= \{z : |z| = |1 + ig(1)|,\ |\arg(z)| \leqslant \arg(1 + ig(1))\}, \\ E &:= E_1 \cup E_2 \cup E_3. \end{aligned} \tag{2.13}$$

It is obvious that E possesses a cusp in the origin. Hence, by virtue of (2.7), E is not quasiconformal.

THEOREM 5. *The assumption on quasiconformality in Theorem* 3 *is essential. More precisely, for every function g as in* (2.10), (2.11) *and* (2.12) *and the corresponding* (*non-quasiconformal curve*) *E as in* (2.13), *there exist constants* $C_1, C_2 > 0$, *depending only on E, and a familiy* $(\sigma_\varepsilon)_{0 < \varepsilon < \varepsilon_0}$ *of signed measures, which are the difference of unit measures on E, such that*

$$\|\sigma_\varepsilon\|^2 \log \frac{\operatorname{diam}(E)}{\omega_+^{-1}((1/8)\, D[\sigma_\varepsilon])} \leqslant C_1 \sqrt{\varepsilon}\, \sqrt{g(\varepsilon)} \log \frac{1}{\varepsilon} \tag{2.14}$$

and

$$(D[\sigma_\varepsilon])^2 \geqslant C_2 \varepsilon, \qquad \forall\, 0 < \varepsilon < \varepsilon_0. \tag{2.15}$$

Thus, an estimate like (2.9) *cannot be proven, since g can be chosen to tend to zero arbitrarily fast.*

3. SOME PROPERTIES OF MODULI OF FAMILIES OF CURVES AND A BASIC LEMMA

We intend to use the notion of a module $m(\Gamma)$ of a family of curves Γ (for definition and properties, cited below, see [1, 3, 19]). This quantity is a conformal invariant and satisfies the following property, known as the comparison principle. Let Γ' and Γ'' be two families of curves. If for any $\gamma' \in \Gamma'$ there exists $\gamma'' \in \Gamma''$ with $\gamma'' \subseteq \gamma'$ (we write $\Gamma' > \Gamma''$), then

$$m(\Gamma') \leqslant m(\Gamma''). \tag{3.1}$$

In addition, we need another property: Let Ω_1 and Ω_2 be two disjoint open sets in $\overline{\mathbb{C}}$ and Γ_1 resp. Γ_2 families of curves in Ω_1 resp. Ω_2. Then for the moduli we have

$$m(\Gamma_1 \cup \Gamma_2) = m(\Gamma_1) + m(\Gamma_2). \tag{3.2}$$

For later use we need to prove the following lemma.

LEMMA 3.1. *Let $E \subset \mathbb{C}$ be a Jordan arc. Dividing E in 3 closed, except for common endpoints disjoint subarcs E_1, L and E_2, where E_1 and E_2 denote the subarcs containing the endpoints of E and* $\operatorname{diam}(E_i) \geqslant \operatorname{diam}(L)$, $i = 1, 2$, *we have*

$$\operatorname{cap}(E_1, E_2) \geqslant \frac{1}{16\pi^2}.$$

Proof. $\Omega := \overline{\mathbb{C}} \backslash \{E_1 \cup E_2\}$ is doubly connected. Hence, there exists a conformal mapping

$$f\colon \Omega \to A_R := \{z \in \mathbb{C} : 1 < |z| < R\}, \tag{3.3}$$

where

$$\operatorname{cap}(E_1, E_2) = \frac{1}{\log R}. \tag{3.4}$$

Let Γ denote the family of all Jordan curves $\gamma \subset \overline{\mathbb{C}}$, which separate E_1 and E_2 (see Fig. 1). It is well known that

$$m(\Gamma) = m(f(\Gamma)) = \frac{1}{2\pi} \log R, \tag{3.5}$$

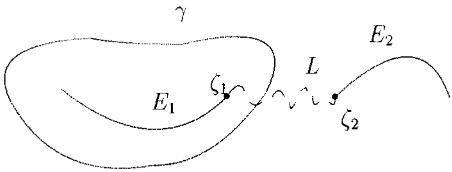

FIGURE 1

where $f(\Gamma) := \{f(\gamma) : \gamma \in \Gamma\}$. Hence, it is enough to find a suitable upper estimate for $m(\Gamma)$. To do so, we need to define some admissible function ρ. Setting $d_i := \operatorname{diam}(E_i)$ for $i = 1, 2$, we choose $\zeta_i := E_i \cap L$ and define

$$\rho_i(z) := \begin{cases} \dfrac{1}{d_i}, & \text{if} \quad |z - \zeta_i| < 2d_i, \\ 0, & \text{elsewhere,} \end{cases}$$

and

$$\rho(z) := \max\{\rho_1(z), \rho_2(z)\}, \qquad \forall z \in \mathbb{C}.$$

It can easily be checked that for all $\gamma \in \Gamma$,

$$\int_\gamma \rho(z)\,|dz| \geqslant 1.$$

Therefore, the definition of the module, (3.4) and (3.5) yield

$$m(\Gamma) \leqslant \int_{\mathbb{C}} \rho^2(z)\,dm(z) \leqslant \int_{\mathbb{C}} \rho_1^2(z)\,dm(z) + \int_{\mathbb{C}} \rho_2^2(z)\,dm(z)$$
$$= \frac{1}{d_1^2} \int_0^{2d_1} r \int_0^{2\pi} d\vartheta\,dr + \frac{1}{d_2^2} \int_0^{2d_2} r \int_0^{2\pi} d\vartheta\,dr = 8\pi,$$

where $dm(z)$ denotes the 2-dimensional Lebesgue measure. ∎

4. PROOF OF THEOREM 1

Throughout this proof we denote by C a positive, absolute constant which can obtain different values in different occurrences.

Using the invariance of the discrepancy and the energy norm of signed measures with total mass 0 with respect to linear transformations we can assume without loss of generality that

$$E \subset \{z \in \mathbb{C} : |z| \leqslant \tfrac{1}{2}\}.$$

We introduce the doubly connected domain $\Omega := \overline{\mathbb{C}} \setminus \{E_1 \cup E_2\}$ and the conformal mapping $f: \Omega \to A_R$ as in (3.3). There exists a continuous extension of f^{-1} to $\overline{A}_R$ for which we assume without loss of generality that $f^{-1}(\{z \in \mathbb{C} : |z| = R\}) = E_2$. For $\varepsilon > 0$ sufficiently small we define the annulus

$$A_{R,\varepsilon} := \{z \in \mathbb{C} : 1 + \varepsilon \leqslant |z| \leqslant R - \varepsilon\}$$

and a function g_1 by

$$g_1(z) := \frac{\log |z| - \log(1+\varepsilon)}{\log(R-\varepsilon) - \log(1+\varepsilon)}, \qquad \forall\, z \in A_{R,\varepsilon}.$$

Then

$$\begin{aligned}
0 \leqslant g_1(z) \leqslant 1, &\qquad \forall\, z \in A_{R,\varepsilon}, \\
g_1(z) = 1, &\qquad \forall\, z \quad \text{with} \quad |z| = R - \varepsilon, \\
g_1(z) = 0, &\qquad \forall\, z \quad \text{with} \quad |z| = 1 + \varepsilon.
\end{aligned}$$

For later use we notice that

$$\begin{aligned}
\int_{A_{R,\varepsilon}} |\text{grad } g_1(z)|^2 \, dm(z) &= \frac{1}{(\log((R-\varepsilon)/(1+\varepsilon)))^2} \int_{A_{R,\varepsilon}} \frac{1}{|z|^2} \, dm(z) \\
&= \frac{2\pi}{\log((R-\varepsilon)/(1+\varepsilon))}.
\end{aligned} \tag{4.1}$$

If $\Omega_\varepsilon := f^{-1}(A_{R,\varepsilon})$, its complement in $\overline{\mathbb{C}}$ consists of two open sets $E_{1,\varepsilon}$ and $E_{2,\varepsilon}$, which contain E_1 and E_2, respectively. We consider in $\overline{\mathbb{C}}$ the function g_2 given by

$$\begin{aligned}
g_2(z) := g_1(f(z)), &\qquad \forall\, z \in \Omega_\varepsilon, \\
g_2(z) := 0, &\qquad \forall\, z \in E_{1,\varepsilon}, \\
g_2(z) := 1, &\qquad \forall\, z \in E_{2,\varepsilon}.
\end{aligned}$$

g_2 is continuous in $\overline{\mathbb{C}}$ and its gradient can be extended continuously from both sides to $\partial\Omega_\varepsilon$. Thus, we have

$$g_2 \in C^1(\Omega_\varepsilon) \qquad \text{and} \qquad g_2 \in C^\infty(E_{i,\varepsilon}), \qquad i=1,2.$$

In choosing a continuously differentiable function $g_3(z)=g_3(|z|)$ with

$$\begin{aligned} 0 \leqslant g_3(z) \leqslant 1, &\qquad \forall\, z \in \mathbb{C}, \\ g_3(z)=1, &\qquad \forall\, |z| \leqslant 1, \\ g_3(z)=0, &\qquad \forall\, |z| \geqslant 2, \end{aligned}$$

we consider the function $g(z):=g_2(z)\, g_3(z)$ which satisfies

$$\begin{aligned} 0 \leqslant g(z) \leqslant 1, &\qquad \forall\, z \in \overline{\mathbb{C}}, \\ g(z)=1, &\qquad \forall\, z \in E_{2,\varepsilon}, \\ g(z)=0, &\qquad \forall\, z \in E_{1,\varepsilon}. \end{aligned} \tag{4.2}$$

The function g can be represented as

$$g(z) = -\frac{1}{2\pi}\int_{\mathbb{C}} \operatorname{grad} g(\zeta)\operatorname{grad}_\zeta \log|z-\zeta|\, dm(\zeta), \tag{4.3}$$

where we use the standard inner product in $\mathbb{R}^2$. We skip the proof of (4.3) for it is only a simple application of Green's formulas.

Using (2.6) and (4.2), (4.3), Fubini's theorem and Hölder's inequality, we obtain

$$\begin{aligned} \frac{1}{8}\, m &\leqslant \left|\int_E g(z)\, d\sigma(z)\right| = \left|\frac{1}{2\pi}\int_E\int \operatorname{grad} g(\zeta)\operatorname{grad}_\zeta \log|z-\zeta|\, dm(\zeta)\, d\sigma(z)\right| \\ &= \left|\frac{1}{2\pi}\int \operatorname{grad} g(\zeta)\operatorname{grad} U(\sigma,\zeta)\, dm(\zeta)\right| \\ &\leqslant \left[\int |\operatorname{grad} g(\zeta)|^2\, dm(\zeta)\right]^{1/2} \left[\frac{1}{4\pi^2}\int |\operatorname{grad} U(\sigma,\zeta)|^2\, dm(\zeta)\right]^{1/2}. \end{aligned} \tag{4.4}$$

In addition, the inequality of Minkowski, (4.1) and the fact that $\int |\operatorname{grad} g_1|^2\, dm$ is a conformal invariant with respect to f yield

$$
\begin{aligned}
&\left[\int |\text{grad } g(\zeta)|^2 \, dm(\zeta)\right]^{1/2} \\
&=\left[\int |g_3(\zeta) \text{ grad } g_2(\zeta) + g_2(\zeta) \text{ grad } g_3(\zeta)|^2 \, dm(\zeta)\right]^{1/2} \\
&\leqslant \left[\int |\text{grad } g_2(\zeta)|^2 \, dm(\zeta)\right]^{1/2} + \left[\int |\text{grad } g_3(\zeta)|^2 \, dm(\zeta)\right]^{1/2} \\
&= \left[\int |\text{grad } g_1(\zeta)|^2 \, dm(\zeta)\right]^{1/2} + C \\
&= \left(\frac{2\pi}{\log((R-\varepsilon)/(1+\varepsilon))}\right)^{1/2} + C \leqslant C \left(\frac{2\pi}{\log((R-\varepsilon)/(1+\varepsilon))}\right)^{1/2}, \qquad (4.5)
\end{aligned}
$$

where the constants $C>0$ are absolute under consideration of Lemma 3.1 and (3.4). Finally, since

$$\int |\text{grad } U(\sigma, \zeta)|^2 \, dm(\zeta) = 4\pi^2 \, \|\sigma\|^2,$$

(see [18, Theorem 1.20]), we obtain with (4.4), (4.5) and letting $\varepsilon \to 0$,

$$\frac{1}{8} m \leqslant C \, \|\sigma\| \left(\frac{2\pi}{\log R}\right)^{1/2}.$$

Hence, (3.4) yields

$$D[\sigma] \leqslant c \, \|\sigma\| \sqrt{\operatorname{cap}(E_1, E_2)}$$

with an absolute constant $c>0$. ∎

5. SOME AUXILIARY FACTS FROM THE THEORY OF QUASICONFORMAL MAPPINGS

Let $E \subset \mathbb{C}$ be a K-quasiconformal curve and F be the corresponding quasiconformal mapping, which maps the unit circle onto E such that $F(\infty) = \infty$ and $F(0) = z_0 \in \operatorname{int}(E)$. Let Φ denote the conformal mapping of the unbounded component $\operatorname{ext}(E)$ of $\overline{\mathbb{C}} \backslash E$ onto

$$\Delta := \{z \in \overline{\mathbb{C}} : |z| > 1\},$$

such that $\Phi(\infty) = \infty$ and $\Phi'(\infty) > 0$. In addition, we consider the conformal mapping φ of $\operatorname{int}(E)$ onto the unit disk D, such that $\varphi(z_0) = 0$ and

$\varphi'(z_0)>0$. Since E is quasiconformal, φ and Φ can be extended to quasiconformal mappings of the extended complex plane $\overline{\mathbb{C}}$ onto itself, such that $\varphi(\infty)=\infty$ and $\Phi(z_0)=0$ [1].

Hence, for later use, we cite the following results for conformal and quasiconformal mappings.

LEMMA 5.1 [4, Lemma 1]. *Let g be a conformal mapping of a region $G_1\subset\mathbb{C}$ onto a region $G_2\subset\mathbb{C}$. Then, for each $z\in G_1$,*

$$\frac{1}{4}\frac{\operatorname{dist}(g(z),\partial G_2)}{\operatorname{dist}(z,\partial G_1)}\leqslant|g'(z)|\leqslant 4\,\frac{\operatorname{dist}(g(z),\partial G_2)}{\operatorname{dist}(z,\partial G_1)}.$$

LEMMA 5.2 [2, Lemma 1]. *Let $\omega=G(z)$ be a $\tilde{K}$-quasiconformal mapping from $\overline{\mathbb{C}}$ onto $\overline{\mathbb{C}}$ with $G(\infty)=\infty$. For $z_j\in\mathbb{C}$ and $\omega_j=G(z_j)$, $j=1,2,3$, the inequality $|\omega_1-\omega_2|\leqslant c_1\,|\omega_1-\omega_3|$ implies*

(a) $|z_1-z_2|\leqslant c_2|z_1-z_3|$,

(b) $|\frac{(z_1-z_3)}{(z_1-z_2)}|\leqslant c_3|\frac{(\omega_1-\omega_3)}{(\omega_1-\omega_2)}|^{\tilde{K}}$

with constants $c_i=c_i(c_1,\tilde{K})$, $i=2,3$.

COROLLARY 5.3. *Since G^{-1} is also a $\tilde{K}$-quasiconformal mapping from $\overline{\mathbb{C}}$ onto $\overline{\mathbb{C}}$ with $G^{-1}(\infty)=\infty$, the inequality $|z_1-z_2|\leqslant c_1\,|z_1-z_3|$ implies*

(a) $|\omega_1-\omega_2|\leqslant c_2\,|\omega_1-\omega_3|$,

(b) $|\frac{(\omega_1-\omega_3)}{(\omega_1-\omega_2)}|\leqslant c_3\,|\frac{(z_1-z_3)}{(z_1-z_2)}|^{\tilde{K}}$

with constants $c_i=c_i(c_1,\tilde{K})$, $i=2,3$.

6. PROOF OF LEMMA 2

In the sequel, we denote by C a positive constant which depends only on K and which can obtain different values in different occurrences.

First, we assume that

$$E\subset\{z\in\mathbb{C}:|z|\leqslant\tfrac{1}{2}\}.$$

Let $\Omega:=\overline{\mathbb{C}}\backslash\{E_1\cup E_2\}$ and $f\colon\Omega\to A_R$ be the conformal mapping as in (3.3). For points ζ' and ζ'' of L_1 with

$$|\zeta'-\zeta''|=\delta,$$

we choose points $\zeta_1 \in L_1$, such that

$$\frac{\delta}{2} \leqslant |\zeta' - \zeta_1| = |\zeta'' - \zeta_1| \leqslant \delta, \tag{6.1}$$

and $\zeta_2 \in L_2$ analogously.

Considering the points $\omega_1 := \Phi(\zeta_1)$, $\omega_2 := \Phi(\zeta_2)$ and $\tilde{\omega}_1 := \varphi(\zeta_1)$, $\tilde{\omega}_2 := \varphi(\zeta_2)$ we define a closed Jordan curve Γ through ζ_1 and ζ_2 as follows.

If $\omega_1 = e^{i\vartheta_1}$, $\omega_2 = e^{i\vartheta_2}$ and $\tilde{\omega}_1 = e^{i\tilde{\vartheta}_1}$, $\tilde{\omega}_2 = e^{i\tilde{\vartheta}_2}$ (where without loss of generality $0 \leqslant \vartheta_1 < \vartheta_2 < 2\pi$, $0 \leqslant \tilde{\vartheta}_1 < \tilde{\vartheta}_2 < 2\pi$), we set

$$\begin{aligned}
\Gamma'_{1,j} &:= \{\omega \in \mathbb{C} : \omega = re^{i\vartheta_j},\ 1 \leqslant r \leqslant 2\}, \qquad j = 1, 2,\\
\Gamma'_{1,3} &:= \{\omega \in \mathbb{C} : \omega = 2e^{i\vartheta},\ \vartheta_1 \leqslant \vartheta \leqslant \vartheta_2\},\\
\Gamma'_1 &:= \Gamma'_{1,1} \cup \Gamma'_{1,2} \cup \Gamma'_{1,3},\\
\Gamma'_{2,j} &:= \{\omega \in \mathbb{C} : \omega = re^{i\tilde{\vartheta}_j},\ \tfrac{1}{2} \leqslant r \leqslant 1\}, \qquad j = 1, 2,\\
\Gamma'_{2,3} &:= \{\omega \in \mathbb{C} : \omega = \tfrac{1}{2}e^{i\vartheta},\ \tilde{\vartheta}_1 \leqslant \vartheta \leqslant \tilde{\vartheta}_2\},\\
\Gamma'_2 &:= \Gamma'_{2,1} \cup \Gamma'_{2,2} \cup \Gamma'_{2,3},
\end{aligned}$$

and $\Gamma_1 := \Phi^{-1}(\Gamma'_1)$, $\Gamma_2 := \varphi^{-1}(\Gamma'_2)$, and $\Gamma := \Gamma_1 \cup \Gamma_2$ (see Fig. 2).

In what follows we want to use this choice of Γ to show that $\frac{1}{R-1}$, which behaves similar to $\operatorname{cap}(E_1, E_2)$, can be estimated from above by $\log \frac{k}{\delta}$ with $k := \operatorname{diam}(E)$. Using this inequality we can prove the right-hand inequality in (2.8).

Let $\tilde{\Gamma} := f(\Gamma)$. We consider the integrals

$$I_1 := \int_\Gamma \frac{|d\zeta|}{\operatorname{dist}(\zeta, \partial\Omega)} \qquad \text{and} \qquad I_2 := \int_{\tilde{\Gamma}} \frac{|d\xi|}{\operatorname{dist}(\xi, \partial A_R)}.$$

To compare both integrals we apply Lemma 5.1 to f, $\Omega_\infty := \Omega \backslash \{\infty\}$ and $f(\Omega_\infty)$ to obtain

$$\begin{aligned}
4I_1 &= 4\int_\Gamma \frac{|d\zeta|}{\operatorname{dist}(\zeta, \partial\Omega_\infty)} = 4\int_{\tilde{\Gamma}} \frac{|d\xi|}{|f'(\zeta)|\operatorname{dist}(\zeta, \partial\Omega_\infty)}\\
&\geqslant \int_{\tilde{\Gamma}} \frac{|d\xi|}{\operatorname{dist}(\xi, \partial f(\Omega_\infty))} \geqslant \int_{\tilde{\Gamma}} \frac{|d\xi|}{\operatorname{dist}(\xi, \partial A_R)} = I_2.
\end{aligned} \tag{6.2}$$

Since $\operatorname{dist}(\xi, \partial A_R) \leqslant \frac{R-1}{2}$ for each $\xi \in \tilde{\Gamma}$, we have

$$I_2 = \int_{\tilde{\Gamma}} \frac{|d\xi|}{\operatorname{dist}(\xi, \partial A_R)} \geqslant \int_0^{2\pi} \frac{d\vartheta}{(R-1)/2} = \frac{4\pi}{R-1}. \tag{6.3}$$

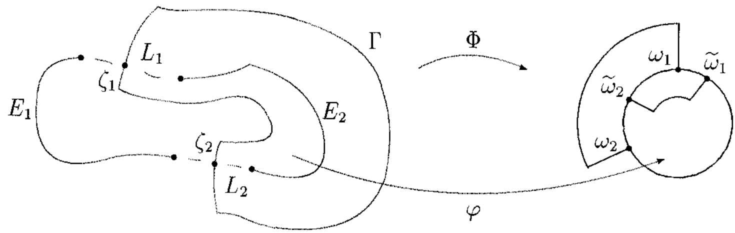

FIGURE 2

With (6.2) and (6.3) it remains to find an upper bound for I_1 in order to give an upper estimate for $\frac{1}{R-1}$. Thus, to consider first

$$\int_{\Gamma_{1,1}} \frac{|d\zeta|}{\operatorname{dist}(\zeta, \partial\Omega)},$$

we need the following lemma.

LEMMA 6.1. *Denoting for each $\zeta \in \Gamma_{1,1}$ the subarc of $\Gamma_{1,1}$ between ζ_1 and ζ with $\Gamma_{1,1}(\zeta_1, \zeta)$, there exist constants $c_i = c_i(K) > 0$, $i = 1, 2$, such that for each $\zeta \in \Gamma_{1,1}$,*

$$c_1 |\zeta - \zeta_1| \leqslant \operatorname{dist}(\zeta, E) \leqslant |\zeta - \zeta_1| \tag{6.4}$$

and

$$|\zeta - \zeta_1| \leqslant l(\Gamma_{1,1}(\zeta_1, \zeta)) \leqslant c_2 |\zeta - \zeta_1|. \tag{6.5}$$

Inequality (6.4) can be proved with standard arguments using the quasiconformality of E. For the proof of (6.5) we refer the reader to [12, p. 48].

To find an upper estimate for the integral above, we define $\omega' := \Phi(\zeta')$ and $\omega'' := \Phi(\zeta'')$. Since

$$|\omega' - \omega_1| \leqslant 2 |\omega_1 - 2e^{i\vartheta_1}|,$$

Lemma 5.2 and (6.1) yield

$$\frac{\delta}{2} \leqslant |\zeta' - \zeta_1| \leqslant c_1 |\zeta_1 - \Phi^{-1}(2e^{i\vartheta_1})| \leqslant c_1 l(\Gamma_{1,1})$$

with $c_1 = c_1(K)$. Hence we can choose a subarc γ_1 of $\Gamma_{1,1}$ of length $\frac{1}{2c_1}\delta$ with ζ_1 as one of its endpoints and we have

$$\int_{\Gamma_{1,1}} \frac{|d\zeta|}{\operatorname{dist}(\zeta, \partial\Omega)} = \int_{\gamma_1} \frac{|d\zeta|}{\operatorname{dist}(\zeta, \partial\Omega)} + \int_{\Gamma_{1,1}\setminus\gamma_1} \frac{|d\zeta|}{\operatorname{dist}(\zeta, \partial\Omega)}.$$

To give an estimate for the first integral of the right-hand side we set $\gamma_1' := \Phi(\gamma_1)$ and assume without loss of generality (the other case can be treated similarly) that

$$|\omega - \omega'| \leqslant |\omega - \omega''|, \qquad \forall\, \omega \in \gamma_1'.$$

Let $\zeta \in \gamma_1$ and $\omega := \Phi(\zeta) \in \gamma_1'$. Defining $L_1' := \Phi(L_1)$, we obtain for each $\tilde{\omega} \in \partial D \setminus L_1'$,

$$|\omega - \omega'| \leqslant |\omega - \tilde{\omega}|.$$

Hence, Lemma 5.2 yields

$$|\zeta - \zeta'| \leqslant C \operatorname{dist}(\zeta, \partial\Omega). \tag{6.6}$$

On the other hand

$$|\omega - \omega'| \geqslant |\omega_1 - \omega'|,$$

and, consequently, using Lemma 5.2 and (6.1) we get

$$\frac{\delta}{2} \leqslant |\zeta_1 - \zeta'| \leqslant C\, |\zeta - \zeta'|. \tag{6.7}$$

The estimates (6.6) and (6.7) imply

$$\operatorname{dist}(\zeta, \partial\Omega) \geqslant C\frac{\delta}{2}, \qquad \forall\, \zeta \in \gamma_1,$$

which leads to

$$\int_{\gamma_1} \frac{|d\zeta|}{\operatorname{dist}(\zeta, \partial\Omega)} \leqslant \frac{C}{\delta}\, l(\gamma_1) \leqslant C. \tag{6.8}$$

To estimate the other integral, Lemma 6.1 yields

$$\int_{\Gamma_{1,1}\setminus\gamma_1} \frac{|d\zeta|}{\operatorname{dist}(\zeta, \partial\Omega)} \leqslant C \int_{\Gamma_{1,1}\setminus\gamma_1} \frac{|d\zeta|}{l(\Gamma_{1,1}(\zeta_1, \zeta))} \leqslant C \log \frac{k}{\delta}. \tag{6.9}$$

Hence, with (6.8) and (6.9) we get

$$\int_{\Gamma_{1,1}} \frac{|d\zeta|}{\operatorname{dist}(\zeta, \partial\Omega)} \leqslant C \log \frac{k}{\delta}.$$

Since $\Gamma_{1,3}$ is a subarc of a level line of the Green's function of $\operatorname{ext}(E)$ with pole at ∞, we have

$$\int_{\Gamma_{1,3}} \frac{|d\zeta|}{\operatorname{dist}(\zeta, \partial\Omega)} \leqslant C.$$

The integral over $\Gamma_{1,2}$ behaves like that over $\Gamma_{1,1}$. Thus, we get

$$\int_{\Gamma_1} \frac{|d\zeta|}{\operatorname{dist}(\zeta, \partial\Omega)} \leqslant C \log \frac{k}{\delta}.$$

Since the case of Γ_2 can be handled in the same way, we obtain

$$I_1 = \int_{\Gamma} \frac{|d\zeta|}{\operatorname{dist}(\zeta, \partial\Omega)} \leqslant C \log \frac{k}{\delta}. \tag{6.10}$$

Consequently, (6.2), (6.3) and (6.10) yield

$$\frac{4\pi}{R-1} \leqslant C \log \frac{k}{\delta}, \tag{6.11}$$

which is the desired estimate for $\frac{1}{R-1}$.

The next step is to find an appropriate lower bound for $\frac{1}{R-1}$. To this end, we define $z_\infty := f(\infty)$ and assume without loss of generality (the other case can be treated similarly) that

$$|z_\infty| \geqslant \frac{R+1}{2}.$$

In setting

$$\tilde{\Gamma} := \left\{ z \in \mathbb{C} : z = \left(1 + \frac{R-1}{8}\right) e^{i\vartheta},\ 0 \leqslant \vartheta \leqslant 2\pi \right\}$$

and $\Gamma := f^{-1}(\tilde{\Gamma})$ we want to make use of the integrals

$$I_1 := \int_{\Gamma} \frac{|d\zeta|}{\operatorname{dist}(\zeta, \partial\Omega)} \qquad \text{and} \qquad I_2 := \int_{\tilde{\Gamma}} \frac{|d\xi|}{\operatorname{dist}(\xi, \partial A_R)}.$$

To compare both integrals we apply Lemma 5.1 to f, $A_R^\infty := A_R \setminus \{z_\infty\}$ and $f^{-1}(A_R^\infty)$ to obtain

$$4I_2 = 4\int_{\tilde{\Gamma}} \frac{|d\xi|}{\operatorname{dist}(\xi, \partial A_R^\infty)} \geqslant \int_{\tilde{\Gamma}} \frac{|d\xi|}{|f'(\zeta)| \operatorname{dist}(\zeta, \partial f^{-1}(A_R^\infty))}$$

$$= \int_{\Gamma} \frac{|d\zeta|}{\operatorname{dist}(\zeta, \partial f^{-1}(A_R^\infty))} = \int_{\Gamma} \frac{|d\zeta|}{\operatorname{dist}(\zeta, \partial\Omega)} = I_1. \tag{6.12}$$

By virtue of the equality

$$I_2 = \frac{8}{R-1}\int_0^{2\pi} \frac{R+7}{8}\, d\vartheta = 2\pi \frac{R+7}{R-1}, \tag{6.13}$$

it remains to find an estimate from below for I_1. In choosing $\zeta_0 \in \Gamma \cap E$ we have for each $\zeta \in \Gamma$,

$$\operatorname{dist}(\zeta, \partial\Omega) \leqslant \operatorname{dist}(\zeta_0, \partial\Omega) + |\zeta_0 - \zeta|$$

$$\leqslant \delta + l(\Gamma(\zeta_0, \zeta)),$$

where $\Gamma(\zeta_0, \zeta)$ denotes a subarc of Γ with endpoints ζ_0 and ζ. All subarcs $\Gamma(\zeta_0, \zeta)$ are chosen with the same orientation. Consequently, we get

$$I_1 \geqslant \int_{\Gamma} \frac{|d\zeta|}{\delta + l(\Gamma(\zeta_0, \zeta))} = \int_0^{l(\Gamma)} \frac{ds}{\delta + s} = \log \frac{\delta + l(\Gamma)}{\delta}.$$

The assumptions on E_1 and E_2 yield $l(\Gamma) \geqslant \frac{1}{6}\operatorname{diam}(E)$ and we obtain

$$I_1 \geqslant C \log \frac{k}{\delta}. \tag{6.14}$$

Finally, (6.12), (6.13) and (6.14) lead to the desired inequality

$$\log \frac{k}{\delta} \leqslant C \frac{R+7}{R-1}. \tag{6.15}$$

Since by Lemma 3.1, R is bounded from above by an absolute constant, we have

$$C \frac{1}{\log R} \leqslant \frac{1}{R-1} \leqslant \frac{1}{\log R}$$

with an absolute constant $C>0$. Hence, (3.4), (6.11), and (6.15) yield

$$C_1 \log \frac{k}{\delta} \leqslant \operatorname{cap}(E_1, E_2) \leqslant C_2 \log \frac{k}{\delta}.$$

It remains to consider the case $\operatorname{diam}(E)>1$. In defining a new quasiconformal curve

$$\tilde{E} := \left\{ \zeta = \frac{z}{\operatorname{diam}(E)} : z \in E \right\}$$

the proof of the first part can be repeated with $\tilde{E}$. But then the desired estimates are also valid for E since shrinking leaves the problem invariant. ∎

7. PROOF OF THEOREM 3

If E is a quasiconformal curve, Theorem 3 follows by combining Theorem 1 and Lemma 2. Therefore, we consider only the case when $E \subset \mathbb{C}$ is a quasiconformal arc. Then, there exists a K-quasiconformal mapping F_1 from $\overline{\mathbb{C}}$ onto $\overline{\mathbb{C}}$ such that $F_1(\infty)=\infty$ and $F_1(E)=[0, 1]$. Let

$$Q := \{ z = x + iy : 0 < x < 1, 0 < y < 1 \}.$$

∂Q is a K_1-quasiconformal curve, i.e., there exists a K_1-quasiconformal mapping F_2 from $\overline{\mathbb{C}}$ onto $\overline{\mathbb{C}}$ with $F_2(\infty)=\infty$ and $F_2(\partial D)=\partial Q$. Setting

$$F_3(z) := F_1^{-1}(F_2(z)), \qquad \forall z \in \overline{\mathbb{C}},$$

we obtain a $K_1 K$-quasiconformal mapping with $F_3(\infty)=\infty$ and $E \subset F_3(\partial D)$. Hence $E' := F_3(\partial D)$ is a $K_1 K$-quasiconformal curve. Considering σ as a signed measure on E', we obtain with Theorem 1 and Lemma 2,

$$D[\sigma] \leqslant c_1 \|\sigma\| \sqrt{\log \frac{2k'}{\omega_+^{-1}((1/8)\, D[\sigma])}}, \tag{7.1}$$

where $k' := \operatorname{diam}(E')$ and $c_1>0$ depends only on K. It remains to show that k' can be replaced by k in (7.1).

It suffices to show that

$$C \operatorname{diam}(E') \leqslant \operatorname{diam}(E) \leqslant \operatorname{diam}(E'). \tag{7.2}$$

Let z_1 and z_2 denote the endpoints of E. We choose $\zeta_1 \in E$ such that

$$|\zeta_1 - z_1| \geqslant \tfrac{1}{2} \operatorname{diam}(E).$$

Since

$$|F_1(\zeta_1) - F_1(z_1)| \leqslant |F_1(z_1) - F_1(z_2)|,$$

Lemma 5.2 implies

$$\operatorname{diam}(E) \geqslant |z_1 - z_2| \geqslant C\,|\zeta_1 - z_1| \geqslant \frac{C}{2} \operatorname{diam}(E).$$

Analogously,

$$C \operatorname{diam}(\overline{E' \backslash E}) \leqslant |z_1 - z_2| \leqslant \operatorname{diam}(\overline{E' \backslash E}).$$

Hence, we obtain (7.2). Finally, (7.1) and (7.2) lead to

$$\begin{aligned} D[\sigma] &\leqslant c_1 \, \|\sigma\| \sqrt{\log \frac{2 \operatorname{diam}(E')}{\omega_+^{-1}((1/8)\, D[\sigma])}} \leqslant c_1 \, \|\sigma\| \sqrt{\log \frac{2Ck}{\omega_+^{-1}((1/8)\, D[\sigma])}} \\ &\leqslant c \, \|\sigma\| \sqrt{\log \frac{2k}{\omega_+^{-1}((1/8)\, D[\sigma])}}. \quad \blacksquare \end{aligned}$$

8. PROOF OF THEOREM 4

Let $E \subset \mathbb{C}$ be some K-quasiconformal curve with $\operatorname{diam}(E) = 1$. We will use condenser theory to construct a family of signed measures $(\sigma_\delta)_{0 < \delta < \delta_0}$ for which (2.9) is sharp up to the constant.

For $0 < m < 1/2$ choose disjoint, closed subarcs $L_1, L_2 \subset E$, such that

$$\operatorname{diam}(L_1) = \operatorname{diam}(L_2) = e^{-8/m} =: \delta, \tag{8.1}$$

and

$$\operatorname{diam}(E_i) \geqslant \tfrac{1}{12} \operatorname{diam}(E), \qquad i = 1, 2,$$

where E_1 and E_2 are the remaining subarcs with $E_1 \cup E_2 := \overline{E \backslash \{L_1 \cup L_2\}}$. The subarcs E_1, E_2 and L_1, L_2 satisfy the assumptions of Lemma 2. Thus, there exist constants $0 < c_1 < 1$ and $c_2 > 0$ only depending on K, such that

$$c_1 \log \frac{1}{\delta} \leqslant \operatorname{cap}(E_1, E_2) \leqslant c_2 \log \frac{1}{\delta}. \tag{8.2}$$

To construct the desired signed measure σ_δ we consider the equilibrium measure $\tau = \tau_{E_1} - \tau_{E_2} \in \mathcal{M}^1(E_1, E_2)$, i.e.,

$$\|\tau\|^2 = \operatorname{mod}(E_1, E_2). \tag{8.3}$$

Defining $\sigma_\delta := \sigma_\delta^+ - \sigma_\delta^-$ by

$$\sigma_\delta^+ := \tfrac{1}{2}(\tau_{E_1} + \tau_{E_2}),$$
$$\sigma_\delta^- := (\tfrac{1}{2} - m)\, \tau_{E_1} + (\tfrac{1}{2} + m)\, \tau_{E_2},$$

we have

$$\begin{aligned} m = D[\sigma_\delta] &= m \sqrt{\operatorname{mod}(E_1, E_2)}\, \sqrt{\operatorname{cap}(E_1, E_2)} \\ &= \|\sigma_\delta\| \sqrt{\operatorname{cap}(E_1, E_2)}. \end{aligned} \tag{8.4}$$

To prove the desired estimate for σ_δ we need to estimate the inverse of the modulus of continuity ω_+ of σ_δ^+ from below. Introducing

$$\omega^{-1}(\tau_{E_j}, t) := \inf\{\operatorname{diam}(J) : J \text{ a subarc of } E_j,\ \tau_{E_j}(J) \geqslant t\},$$

for $j = 1, 2$, we have

$$\omega_+^{-1}(t) \geqslant \min\{\omega^{-1}(\tau_{E_1}, t), \omega^{-1}(\tau_{E_2}, t)\}, \qquad \forall\, 0 < t \leqslant 1. \tag{8.5}$$

We claim that

$$\log \frac{1}{\omega_+^{-1}((1/8)\, m)} \leqslant c_3 \operatorname{cap}(E_1, E_2) \tag{8.6}$$

for some $c_3 = c_3(K) > 0$, so that (8.4) and (8.6) yield

$$m = D[\sigma_\delta] \geqslant c\, \|\sigma_\delta\| \sqrt{\log \frac{1}{\omega_+^{-1}((1/8)\, m)}}.$$

for some $c = c(K) > 0$. Consequently, if (8.6) holds, inequality (2.9) is sharp up to the constant for σ_δ.

To prove (8.6) it is enough to give appropriate lower bounds for $\omega^{-1}(\tau_{E_1}, \frac{1}{8} m)$ and $\omega^{-1}(\tau_{E_2}, \frac{1}{8} m)$. Since the other case can be treated similarly, we will without loss of generality derive the lower bound for $\omega^{-1}(\tau_{E_1}, \frac{1}{8} m)$ only. Let $J \subset E_1$ be some closed subarc and Γ be the family of all Jordan arcs in $\Omega := \overline{\mathbb{C}} \setminus \{E_1 \cup E_2\}$ with endpoints on J and E_2 (see Fig. 3). By giving upper and lower estimates for the module $m(\Gamma)$ we will obtain the desired result.

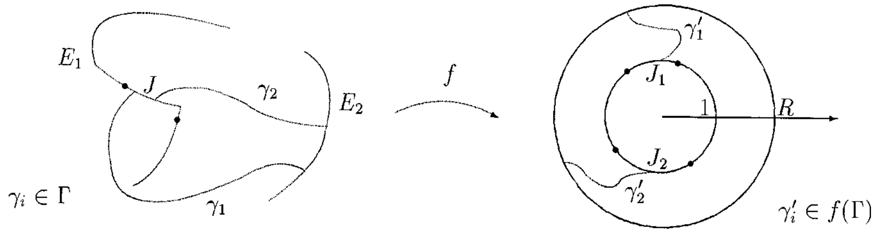

FIGURE 3

Let $f: \Omega \to A_R$ denote the conformal mapping of (3.3). Without loss of generality we assume that

$$f^{-1}(E_1) = \{z : |z| = 1\}.$$

If J_1 and J_2 denote the two preimages of J with

$$J_1 = \{e^{i\vartheta} : \vartheta_{1,1} \leqslant \vartheta \leqslant \vartheta_{1,2}\},$$
$$J_2 = \{e^{i\vartheta} : \vartheta_{2,1} \leqslant \vartheta \leqslant \vartheta_{2,2}\},$$

(2.2), (3.1), (3.2) and an example in [3] yield

$$\begin{aligned} m(\Gamma) = m(f(\Gamma)) &\geqslant m(\{\{re^{i\vartheta} : 1 < r < R\} : \vartheta_{1,1} \leqslant \vartheta \leqslant \vartheta_{1,2}\}) \\ &\quad + m(\{\{re^{i\vartheta} : 1 < r < R\} : \vartheta_{2,1} \leqslant \vartheta \leqslant \vartheta_{2,2}\}) \\ &= \frac{\vartheta_{1,2} - \vartheta_{1,1}}{\log R} + \frac{\vartheta_{2,2} - \vartheta_{2,1}}{\log R} \\ &= \frac{2\pi\tau_{E_1}(J)}{\log R}. \end{aligned} \tag{8.7}$$

To find an upper bound for $m(\Gamma)$ consider two points $z_1 \in E_1$ and $z_2 \in E_2$, such that

$$\operatorname{dist}(E_1, E_2) = |z_1 - z_2|.$$

Since E is quasiconformal, (2.7) and (8.1) yield

$$|z_1 - z_2| \geqslant c_4 \operatorname{diam}(L_1) = c_4 \delta$$

with $c_4 = c_4(K) > 0$. Next, we restrict J such that

$$\delta_1 := \operatorname{diam}(J) < \frac{1}{e^{1/c_1}} c_4 \delta,$$

and define for some $z_3 \in J$,

$$A_{z_3} := \{z \in \mathbb{C} : \delta_1 < |z - z_3| < c_4 \delta\}.$$

Let Γ_1 denote the family of all Jordan arcs in A_{z_3} which connect its boundary elements. Then $\Gamma > \Gamma_1$ and, consequently, with (3.1),

$$m(\Gamma) \leqslant m(\Gamma_1) = \frac{2\pi}{\log(c_4 \delta/\delta_1)} < c_1 2\pi. \tag{8.8}$$

Now, by virtue of (8.1), (8.2), (8.7) and (8.8) we obtain

$$\tau_{E_1}(J) < c_1 \log R = c_1 \frac{1}{\operatorname{cap}(E_1, E_2)} < \frac{1}{\log(1/\delta)} = \frac{m}{8},$$

and, therefore,

$$\omega^{-1}\left(\tau_{E_1}, \frac{m}{8}\right) \geqslant c_5 \delta \tag{8.9}$$

with $c_5 = c_5(K) > 0$. Finally, Lemma 3.1, (8.2), (8.5), and (8.9) yield

$$\log \frac{1}{\omega_+^{-1}((1/8)\, m)} \leqslant \log \frac{1}{c_3 \delta} \leqslant c_3 \operatorname{cap}(E_1, E_2),$$

i.e., (8.6) holds.

Finally, we remark that $\delta = e^{-8/m}$ was chosen arbitrarily with $0 < m < 1/2$. Hence, inequality (2.9) is sharp up to the constant for the family of signed measures $(\sigma_\delta)_{0<\delta<\delta_0}$ with $\delta_0 := e^{-16} = e^{-8/m_0}$ for $m_0 := 1/2$. In addition, the right-hand inequality in (8.2) yields

$$\frac{D[\sigma_\delta]}{\|\sigma_\delta\|} \to 0 \qquad \text{as} \qquad \delta \to 0,$$

i.e., (2.9) is sharp up to the constant. ∎

9. PROOF OF THEOREM 5

In the following we denote by $C > 0$ a constant depending only on E, which obtains different values at different occurrences.

Let g be some function with (2.10), (2.11) and (2.12) and let the Jordan curve E be defined as in (2.13). For $\varepsilon_0 > 0$ sufficiently small, we choose some $0 < \varepsilon < \varepsilon_0$ and consider the subarcs

$$E^+ := E \setminus \{x + ig(x) : x \in (0, \varepsilon)\},$$

$$E^- := E \setminus \{x - ig(x) : x \in (0, \varepsilon)\},$$

and define $\sigma_\varepsilon := \mu_{E^+} - \mu_{E^-}$, where μ_{E^+} resp. μ_{E^-} denote the equilibrium measures of E^+ resp. E^- [24, p. 55].

To derive the inequalities (2.14) and (2.15) we need to prove some helpful estimates. First, we show that for each

$$z \in E_\varepsilon := \{x + ig(x) : x \in [0, \varepsilon]\},$$

there holds

$$|U(\mu_{E^+} - \mu_{E^-}, z)| \leqslant c_0 \sqrt{g(\varepsilon)} \tag{9.1}$$

with $c_0 = c_0(E) > 0$. Let

$$\Phi_+ : \overline{\mathbb{C}} \setminus E^+ \to \Delta$$

be the conformal mapping such that $\Phi_+(\infty) = \infty$ and $\Phi'_+(\infty) > 0$. Using the fact, that $U(\mu_{E^+}, z)$ and $U(\mu_{E^-}, z)$ are constant on E^+ and E^-, respectively, and using the symmetry of E^+ and E^- we get

$$U(\mu_{E^+} - \mu_{E^-}, z) = -\log |\Phi_+(z)|, \qquad \forall\, z \in E_\varepsilon. \tag{9.2}$$

To get an estimate for $\Phi_+(z)$ with $z \in E_\varepsilon$, we consider the level lines of Φ_+, which are defined as

$$L_u := \{z \in \overline{\mathbb{C}} \setminus E^+ : |\Phi_+(z)| = 1 + u\}, \qquad u \geqslant 0.$$

We have

$$\operatorname{dist}(E^+, L_u) \geqslant Cu^2$$

(see [3, Corollary 2.7]). Supposing that there exists $z \in E_\varepsilon$ such that

$$|\Phi_+(z)| > 1 + \sqrt{\frac{2g(\varepsilon)}{C}}, \tag{9.3}$$

we would have

$$\operatorname{dist}(z, E^+) > 2g(\varepsilon),$$

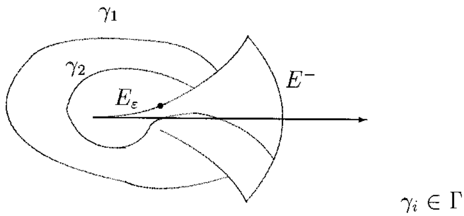

FIGURE 4

which is not possible. Hence (9.2) and the fact, that inequality (9.3) is wrong for each $z \in E_\varepsilon$, yield

$$|U(\mu_{E^+}, z) - U(\mu_{E^-}, z)| = \log |\Phi_+(z)| \leqslant \log\left(1 + \sqrt{\frac{2g(\varepsilon)}{C}}\right) \leqslant c_0 \sqrt{g(\varepsilon)},$$

i.e., (9.1) holds.

Next, we show that there exist positive constants c_1 and c_2, only depending on E, such that

$$c_1 \sqrt{\varepsilon} \leqslant \mu_{E^-}(E_\varepsilon) \leqslant c_2 \sqrt{\varepsilon}. \tag{9.4}$$

Let Γ denote the family of all Jordan arcs in $\overline{\mathbb{C}} \backslash E^-$ with endpoints on E^- which separate E_ε from ∞ (see Fig. 4).

If Φ_- denotes the conformal mapping of $\overline{\mathbb{C}} \backslash E^-$ onto Δ such that $\Phi_-(\infty) = \infty$ and $\Phi'_-(\infty) > 0$, there are two preimages E_ε^1 and E_ε^2 of E_ε. Defining $E'_\varepsilon := E_\varepsilon^1 \cup E_\varepsilon^2$, (2.1) yields

$$\mu_{E^-}(E_\varepsilon) = \frac{1}{2\pi} l(E'_\varepsilon),$$

and considering the inequalities (see [3])

$$\frac{1}{\pi} \log \frac{2}{l(E'_\varepsilon)} \leqslant m(\Gamma) = m(\Phi_-(\Gamma)) \leqslant 2 + \frac{1}{\pi} \log \frac{4}{l(E'_\varepsilon)}$$

we obtain

$$\frac{1}{\pi} \log \frac{1}{\pi \mu_{E^-}(E_\varepsilon)} \leqslant m(\Gamma) \leqslant \frac{1}{\pi} \log \frac{2e^{2\pi}}{\pi \mu_{E^-}(E_\varepsilon)}. \tag{9.5}$$

In proving other upper and lower estimates for $m(\Gamma)$ we will be able to show (9.4). Let

$$A := \{z \in \mathbb{C} : |\varepsilon + ig(\varepsilon)| \leqslant |z| \leqslant 1\}$$

and let Γ_1 denote the family of all Jordan curves in A, which separate the boundary parts. Since $\Gamma_1 > \Gamma$ and since ε_0 was chosen sufficiently small, we have with (3.1) and (2.11),

$$m(\Gamma) \geqslant m(\Gamma_1) = \frac{1}{2\pi} \log \frac{1}{|\varepsilon + ig(\varepsilon)|} \geqslant \frac{1}{2\pi} \log \frac{1}{2\varepsilon}, \tag{9.6}$$

and obtain from (9.5) and (9.6) that

$$\frac{1}{\pi} \log \frac{1}{\sqrt{2\varepsilon}} \leqslant \frac{1}{\pi} \log \frac{2e^{2\pi}}{\pi \mu_{E^-}(E_\varepsilon)}.$$

Consequently,

$$\mu_{E^-}(E_\varepsilon) \leqslant c_2 \sqrt{\varepsilon}$$

with some absolute constant $c_2 > 0$.

To prove the left-hand side inequality in (9.4) we consider the family Γ_2 of all Jordan arcs in $\operatorname{ext}(E)$ with endpoints on E which separate E_ε and ∞. Since $\Gamma > \Gamma_2$,

$$m(\Gamma_2) \geqslant m(\Gamma). \tag{9.7}$$

If Φ denotes the conformal mapping of $\operatorname{ext}(E)$ onto Δ such that $\Phi(\infty) = \infty$ and $\Phi'(\infty) > 0$, we have under consideration of [20, Theorem 3.9],

$$m(\Gamma_2) = m(\Phi(\Gamma_2)) \leqslant 2 + \frac{1}{\pi} \log \frac{4}{l(\Phi(E_\varepsilon))} \leqslant 2 + \frac{1}{\pi} \log \frac{C}{\sqrt{\varepsilon}}. \tag{9.8}$$

Finally, (9.5), (9.7) and (9.8) yield

$$\frac{1}{\pi} \log \frac{1}{\pi \mu_{E^-}(E_\varepsilon)} \leqslant \frac{1}{\pi} \log \frac{C}{\sqrt{\varepsilon}},$$

i.e.,

$$c_1 \sqrt{\varepsilon} \leqslant \mu_{E^-}(E_\varepsilon)$$

holds with $c_1 = c_1(E) > 0$.

In applying the same methods which we used to prove the right-hand inequality of (9.4) we get for each subarc $J \subset E^-$,

$$\mu_{E^-}(J) \leqslant C \sqrt{\operatorname{diam}(J)}.$$

Hence, because of the symmetry of E^+ and E^-,

$$\omega_+^{-1}(t) = \inf\{\operatorname{diam}(J) : J \text{ a subarc of } E, \mu_{E^+}(J) \geqslant t\} \geqslant Ct^2, \qquad \forall\, 0 < t \leqslant 1. \tag{9.9}$$

Now, (9.1), (9.4) and (9.9) yield

$$\begin{aligned} \|\sigma_\varepsilon\|^2 &= \int_E U(\mu_{E^+} - \mu_{E^-}, z)\, d(\mu_{E^+} - \mu_{E^-})(z) \\ &= 2 \int_{E_\varepsilon} |U(\mu_{E^+} - \mu_{E^-}, z)|\, d\mu_{E^-}(z) \\ &\leqslant C \sqrt{\varepsilon} \sqrt{g(\varepsilon)}. \end{aligned} \tag{9.10}$$

Finally, we obtain with (9.4)

$$(D[\sigma_\varepsilon])^2 \geqslant C_2 \varepsilon$$

with $C_2 := c_1^2 > 0$. On the other hand, (9.9) and (9.10) yield

$$\|\sigma_\varepsilon\|^2 \log \frac{\operatorname{diam}(E)}{\omega_+^{-1}((1/8)\, D[\sigma_\varepsilon])} \leqslant C_1 \sqrt{\varepsilon} \sqrt{g(\varepsilon)} \log \frac{1}{\varepsilon},$$

for some $C_1 = C_1(E) > 0$. ∎

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